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Mathematical Methods in Dynamic Economics

András Simonovits

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Mathematical Methods in Dynamic Economics

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FOREWORD

Almost from the beginning of my research career (1970), I have been working on Dynamic Problems. Since 1988, I have been teaching economic dynamics at the Budapest University of Economics (called Karl Marx University of Economics till 1990). After several years of experience, I realized that the students need a relatively short lecture note on the mathematical methods of the dynamic economics. Or turning the idea upside down: I have to present those dynamic economic models where the most important mathematic methods are applicable.

In 1993 the College László Rajk (Budapest University of Economics) accepted my proposal to teach a one-year course on the subject. The financial help of the Hungarian Soros Foundation was also helpful. Since 1993 I have also been giving a revised course to the participants of the Ph.D. Program for Economics at that university.

The critical remarks of the participants of these courses have contributed to the improvement on the original material. In the Spring of 1995 I repeated these lectures at the Department of Economics, University of Linz. For that occasion I translated Part I of the notes into English. Since then the Hungarian version has been published by the KJK (Publisher of Economics and Law), and the present English version is an abridged variant of the former.

other hand, I forsake the sophisticated proofs which use deep results of Functional Analysis, Measure Theory and other higher mathematical fields. As a result, my approach is more demanding than that of Chiang (1984) but less demanding than that of Stokey and Lucas (1989). Only the reader can decide if this trial is successful or not.

By choosing among the innumerable dynamic economic models, I wanted to present classic models like Hicks' trade cycle model and Samuelson's price dynamics. On the other hand, I also wished to present fresh material, including some of my own works. I only hope that the reader will accept this mix.

At my permanent affiliation (Institute of Economics, Hungarian Academy of Sciences) as well as at my visiting posts, a good deal of colleagues have influenced me. At the first place I mention János Kornai, who introduced me into the topics *Non-Price Control* and *Macroeconomics of the Shortage Economy.* (Unfortunately, papers dealing with the later topic had no place in the English version.) Our long cooperation is reflected not only in our joint publications but in other, subsequent works: both types will be reported in the book. Cars Hommes and Helena Nusse have taught me the modern chapters of nonlinear dynamics during a joint research (which has also been dropped from the English version). György Molnár was a most valuable co-author in modelling *Overlapping Cohorts.*

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Budapest, April 1999.

INTRODUCTION

In this book, we shall deal with relatively simple dynamic economic models. The presentation of each model is preceded by the discussion of the mathematical prerequisites. More precisely, the subsequent mathematical and economic chapters form pairs: each mathematical chapter (with odd number) prepares the ground for the subsequent economic chapter (with even number). There are exceptions at the end of the book: Appendix A contains some auxiliary material from Linear Algebra and Appendices B and C are devoted to additional economic material. Of course, both mathematical and economic chapters are built on their previous counterparts. This structure separates the mathematical groundwork from economic models, but requires the reader to study mathematical problems without knowing their economic use.

The book does not aim at completeness. The basic objective is to present some important methods and models of economic dynamics. The table of content gives an idea on the content of the book, therefore in this introductory chapter I will try only to underline the main characteristics of the book.

PROBLEMS

In the Introduction we shall first overview the most important problems: statics versus dynamics, discrete or continuous time, optimize or not, control-theoretical framework, stability and viability, linearity versus nonlinearity, deterministic versus stochastic models, consumers with finite or infinite lifespan, aggregate versus disaggregate models, elegance versus relevance. Some important ideas are taken from Kornai and Martos (1981a).

Statics versus dynamics

Traditional mathematical economics is *static*: there are no two variables with different time-periods which enter the same equation. For example, in the basic model of general equilibrium (Arrow and Debreu, 1954) the vectors of demand, supply and price refer to a single time-period. There are *quasi-dynamic* models where variables with different time-periods are connected, but in a trivial way (for example, the equilibrium volume and price paths of the Neumann model, (Neumann, 1938)). The real economy, however, is *dynamic*: variables of different time-periods are related to each other in an essential way (Frisch, 1933).

Discrete or continuous time

The duality of discrete and continuous time plays an outstanding role in both mathematics and economics. The daily and physical concepts of time favor the continuous approach, while the tools of measurement and calculations prefer the discrete method. The mathematical theories of time-processes generally work with continuous time. Their basic tools are the *differential equations.* Time is the independent variable, and the dependent variables and their derivatives enter the same system of equations. For example, in the differential equation of the mathematical pendulum, the acceleration (that is, the second derivative of the position) is approximately proportional to the position where the proportionality factor is negative.

In contrast, numerical analysis inevitably applies discrete time (steps), and relies on *difference equations.* Time is again the independent variable, but now the dependent variables and their differences or their lagged values enter the same system of equations. For example, in case of annual capitalization, capital and its increment are connected by the interest rate.

On the one hand, continuous-time systems are more complex than discrete-time systems, since the existence and the uniqueness of a solution to differential equations are already deep mathematical results. On the other hand, continuous-time systems are less complex than discrete-time systems, since the resulting paths are simpler in the former than in the latter. For example, in the scalar case no cycle can arise, in the planar case chaos is excluded in the former but not in the latter. It is of particular interest that the analysis of continuous-time systems can be approximated by discrete-time systems when the length of the time-period goes to zero. Among the first economic models we find both continuous and discrete-time models. For example, Samuelson (1939a) modelled the accelerator–multiplier interaction with a discrete-time framework, while Samuelson (1941) described the dynamic adjustment of prices by a continuous-time model. Moreover, the first business cycle models used mixed (difference–differential) equations (for example, Frisch, 1933). Since the pioneers appeared in the scene, both approaches have had extended literature. As a consequence, this book also follows both styles. However, it would be inappropriate to work out both frameworks at full length. Depending on the economic content we shall present sometimes one, sometimes the other method. (If I had to choose between them in economic applications, I would select the discrete framework, because it gives a better approximation to economic problems more often than its continuous counterpart.)

Optimize or not?

The first dynamic economic models (the Walrasian price dynamics by Samuelson, the Hicksian trade cycle model and so on) were not based on optimization. It took several decades that Ramsey (1928) was followed by modern dynamic optimization models (for example, Tinbergen, 1960). The models without optimization are presently unpopular, since the decision-makers' behavior is not derived from first principles. It is of interest that in physics it is useful to derive the objective laws of motion from the maximization of certain functions, although nobody thinks that anybody executes this otherwise meaningless optimization.

The 'genuinely modern' authors simply exclude models without optimization from the realm of mathematical economics. For example, commenting on Solow's realistic assumption of time-invariant saving ratio in his famous neoclassical growth model, Azariadis (1993, p. 4) wrote: "Solow made an *ad hoc* assumption – and there are few sins as grave as this for a self-respecting economist."

Some 'moderately modern' books take a balanced view concerning optimization. Blanchard and Fischer (1989, p. 28) argue as follows: "Our neoclassical bent does not extend to thinking that the only valid macroeconomic models are those explicitly based on maximization. ...[W]e believe that waiting for a model based on first principles before willing to analyze current events and give policy advice is a harmful utopia that leaves the real world to the charlatans rather than to those who recognize the uncertainties of our current knowledge."

The present author is indifferent whether the behavioral assumptions are stylized facts of real life or derived from optimization under constraints. (It is ironic that a number of economists use the distinction *descriptive* versus *normative* approaches and clearly prefer the latter to the former.) As a defense of the nonoptimizing approach, I mention three factors: (i) Historical interest mentioned above. (ii) Most dynamic optimization models have only one decision-maker and it is questionable whom the *representative agent* represents (Kirmann, 1992). Anyway, critique on the exaggerated role of optimization in economics is still relevant (Kornai, 1971, Nelson and Winter, 1982 and Anderson et al., 1988). (iii) The insistence on optimization makes the mathematical treatment more difficult than otherwise, preventing to follow the principle *first things first.*

Correspondingly, the material is divided into two parts: Parts I and II deal with economic dynamics without and with optimization, respectively. For example, the control of a multisector economy would be difficult to describe as optimal strategies of decision-makers, especially a single one. On the other hand, to understand life-cycle problems of one person or the whole society's consumption path, the application of intertemporal optimization may be relevant.

Control-theoretic framework

The book often uses the control-theoretic framework. We shall speak of a *control system* if the variables of a dynamical system are divided into two groups: *state* and *control* variables. We are given a *state-space equation*, which describes the feasible dynamics of the *state vector* as a function of the *control vector*. Perhaps the simplest control is the *feedback* when the control vector only depends on the current state vector. Among other advocates of the control theoretical approach, Kornai and Martos (1981a, b) convincingly argue for the advantages of this approach. As an illustration, we shall choose a leading example from the book edited by them: the change in the output stock of a product is equal to output less sales. In the simplest *control by stock signal*, the output is a decreasing function of the output stock (see for example, Section 2.2). In the bulk of the economic models, *decentralization* is much emphasized. To generalize the previous example: suppose that the whole economy is controlled by stock signals, then the change in the output stock of a firm is equal to its output less total sales, while its output is still a decreasing function of its own output stock. At the same time, the macrobehavior rules are not decentralized.

Notwithstanding its advantages, the control-theoretic approach is not exclusive in the dynamic economics. For example, in the closed exchange models of Overlapping Generations and Cohorts (Appendices B and C) the potential state variable (the accumulated saving) is fixed at zero, thus control theory is not applicable.

Stability and viability

It occurs very frequently that the path of the dynamical system cannot or need not be explicitly determined. This is not always a serious problem, because we are often interested in the qualitative rather than the quantitative behavior of the system. The methods described below help answer our questions.

As a starting point, let us consider the *fixed points*, that is, the rest points of the system: if a path starts from there, it stays there forever. In natural sciences a fixed point is frequently called an *equilibrium point*, or simply, an *equilibrium.* In economic applications, the notion of equilibrium is often restricted to that of *walrasian* equilibrium (see Section 6.3 below) where the self-regulating market mechanism ensures the equality of demand and supply. Since Keynes (1936) economists have also been speaking of *unemployment equilibrium* and since the 1970's of *nonwalrasian equilibrium.* Kornai (1980) and Kornai and Martos (1981b) prefer the more neutral expression *normal state.* Since in Overlapping Generations and Cohorts models (Appendices B and C) certain nonstationary paths are also called equilibrium path, we shall also use the expression *steady state* for a stationary path.

The following double question arises: does there exist a fixed point, and if yes, is it uniquely determined? We shall see that three answers are possible: zero, one and multiple equilibria exist. A further question is the *stability* of the equilibrium: if the system does not start from an equilibrium, will it converge to it? More precisely: how large is the basin of attraction of initial points from where the system converges to a fixed point? With a little imprecision: if the basin is large, then the system is globally stable; if it is small, then the system is only locally stable.

In the Nature as well as in the Society *cyclic* processes arise rather frequently. For example, the Earth revolves around the Sun, the seasons follow each other, the human heart beats 60–90 per minutes, the growth rate of the economy oscillates more or less regularly. It is of interest that for every system with a cycle, there exists another system where the states of the cycle of the original system are fixed points. Staying with deterministic systems, stable equilibria and cycles are joined by more complex paths to be called *chaotic.* Then the path depends sensitively on the initial conditions, thus no forecast is possible. Weather is the best-known example of chaos, but it is conceivable that exchange rates also behave chaotically.

In all the three cases one may ask: is the system *viable*? For example, the solar system may function for 10–20 billion years, a human being may live for a hundred years, social systems may survive decades, centuries or millennia. If we study a special economic system, viability means that, in addition to the governing equations, the system satisfies certain conditions as well. In economics the viability conditions are most often nonnegativity conditions. For example, the output cannot be negative. In the model of control by stock signal, inventories are nonnegative but below capacity. We still know little on economic viability and often we have to be content with searching for stability.

Linear versus nonlinear systems

In every mathematical investigation linearity is of paramount interest. With some simplification, a model is homogeneously linear if doubling the values of the input variables doubles those of the output variables. For example, the equation of change in inventory is linear: doubled output and sales imply doubled change in inventories. At first sight the following rule of control by stock signal is also linear: every day at most 100 units are produced, but this maximum is diminished by the double of the end stock of previous day. But what happens if 51 units remained yesterday? Shall we produce –2 units? No. Rather a natural floor (zero lower bound) enters and linearity disappears.

The theory of *finite-dimensional linear systems* is complete. We know the general solution of the system, and with its help we can obtain a lot of quantitative and qualitative results. In this case the local behavior around the equilibrium also determines the global behavior. As a result, cyclic behavior prevails only for exceptional values of parameters (and it has knife-edge behavior). Similarly, unstable behavior implies inviability at least in the long run.

The situation is totally different with *nonlinear systems.* Everything goes already for one-variable nonlinear (discrete-time) systems. For example, local stability can coexist with the lack of global stability; stable cyclic behavior can be observed for a large domain of the parameter space; instability may be consistent with long run viability. As discussed above, in addition to regular *fixed points* and *cyclic* paths, irregular *chaotic* paths also appear. Here we have to rely often on computer simulation even in the two-variable case. Both the mathematicians and the economists had been content with the study of linear or approximately linear systems until quite recently. Only the last decades have witnessed the blossoming of global analysis of nonlinear and unstable dynamical systems. Of course, even in the investigation of such complex systems, the linear system remains the starting point. The book deals with both linear (for example, Chapters 1 and 2) and nonlinear systems (for example, Chapters 3 and 4).

Deterministic versus stochastic models

For modern researchers, the need to study both deterministic and stochastic systems is obvious. It suffices to refer to the coexistence of the classical and the quantum mechanics.

The spread of stochastic approach in dynamic economics is mostly connected to the econometric method. Following the statistical approach, at the estimation of equations it is appropriate to assume stochastic errors. It is small wonder that stochastic optimization plays an outstanding role in the control models based on stochastic equations. These issues are dealt with in Chapter 7. It is noteworthy that in modern dynamic economics the mainstream economists assume that the deterministic core of the economy is stable, and the economy is destabilized *only* by the stochastic shocks. I do not share this view. Following the example of a much smaller but still significant group of mathematical economists, I prefer a cycle theory which is based on deterministic nonlinearities and which also lead to complex dynamics. As a result, in this book stochastic models are underemphasized, nonlinearity is overemphasized, at least with respect to the